$11 \to 12$

2017-09-20.4
2017-09-25.2 → 2017-09-25.1

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CS 598 Pseudorandomness

CS 598 Pseudorandomness : Lecture 8

admin: ps1 return 9/27

ps2 extension: due 10/2

ps3 out 9/27

today: defn of expanders

existence

relations

vertex expanders:

$G = (V, E)$  $N(S) = \{u : (u,v) \in E, v \in S\}$

want: $|E| \le O(|V|)$ [sparse graph] [edges are independent], e.g. D-regular $D \le O(1)$

- "well connected"

defn: G is (K, A) vertex expander if $\forall S \subseteq V$, $|S| \le K$

$|N(S)| \ge A \cdot |S|$

parameters: D-regular graph, $D = O(1)$

$N = |V| \to \infty$ [gen family]

$A \le D$, $KA \le N$

want: $(\underbrace{\Omega(N)}, \underbrace{1+}\ \underbrace{\Omega(1)})$ $\underbrace{O(1)}$ regular vertex expander

hardest regime, most interesting

trivial

about as big as possible

nontrivial, [better given]

random D-regular graphs are

thm: $D \ge 3$, $\exists$ $(\Omega_D(N), D - 1.01)$ D-regular vertex expander w/p $\ge 1/2$

Rmk: - $D = 2$ has no expander [only]

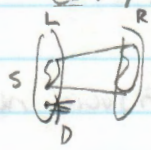
G = union of cycles and paths

- $A = D - 1 - \varepsilon$ is optimal

- random D-regular graphs are complicated [hard to sample?] [hard many]

thm: $D \ge 3$, $\exists \alpha > 0$ st random D-left regular bipartite graphs on $2N$ vertices are $(\alpha N, D-2)$ left-vertex expanders w/p $\ge 1/2$

eg



Pf: prob method

random graph = each vertex in L chooses D neighbors in R, independently

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$k \le \alpha N$, $S \subseteq L$ $|S| = k$ [want S to expand]

choose $N(S)$ in order $v_1, \dots, v_{kD} \in R$ [$N(S)$ small = collisions = repeats]

v_i repeat if $v_i = v_j$ $j < i$

$\Pr[v_i \text{ repeat}] \le \frac{i-1}{N} \le \frac{kD}{N}$ [why]

$\Pr[|N(S)| \le (D-2)|S|] = \Pr[\ge 2k \text{ repeats in } v_1, \dots, v_{kD}]$

$\le \binom{kD}{2k} \cdot \left(\frac{kD}{N}\right)^{2k}$

$\Pr[\exists S \in \binom{[N]}{k}] \le \binom{N}{k} \cdot \le \left(\frac{Ne}{k}\right)^k \left(\frac{kDe}{2k}\right)^{2k} \left(\frac{kD}{N}\right)^{2k}$

$\Pr[\exists k] \le \sum_{k=1}^{\lfloor \alpha N \rfloor} 1/4^k \le 1/3$ $= \left(\frac{e^3 D^4 k}{4N}\right)^k \le 1/4^k$

$k \le \alpha N \le$ ∎

$\alpha := \frac{1}{e^3 D^4}$

Rmk. - biregular vertex expanders via unions of D random perfect matchings [lose independence, but not an issue]

- can translate some results to uniformly random D-regular graphs [via transfer / contiguity thms]

Q. explicit vertex expanders? [helpful to consider other form of expansion]

= Questions?

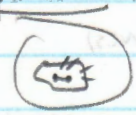
spectral expanders. $G = (V, E)$ D-regular, random walk M

def. $\lambda(G) = \max_{\substack{x \perp U \\ \|x\| = 1}} \|Mx\| \in [0, 1]$ [max 2nd eigenvalue in abs value]

spectral gap: $\gamma(G) = 1 - \lambda(G)$ [spectral expander]

goal. $\gamma(G) \ge \Omega(1)$ [any D, even super constant]

edge expander.

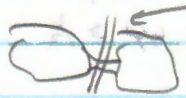


def: G is (k, ε) edge expander if $\forall S \subseteq V$ $|S| \le k$

$|\text{cut}(S, \bar{S})| \ge \varepsilon \cdot D \cdot |S|$ [ε-fraction of edges leave S]

[no bottleneck in graph]

eg.



sparse cut

computing sparse cuts: detect communities in social networks

clustering / partitioning

dynamic programming [limited interaction between subproblems]

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relations:

Q: edge vs vertex vs spectral? [approx edge, vertex by computing spectral]

A: different, yet similar

computing: NP-hard NP-hard in P

params: can give strong relationships

yet: $(N/4, 1+1/4)$ vertex expander $\not\Rightarrow$ $\Omega(1)$ spectral expander

↖ can be disconnected ↳ must be connected

eg disjoint union of expanders

optimal param:

vertex: $(\alpha_D N, D-1, 1)$

spectral $\lambda(G) \le 2\frac{\sqrt{D-1}}{D}$

Fact: random graphs have $\lambda(G) \le 2\frac{\sqrt{D-1}}{D} + o(1)$ [hard stuff, hard to show random graphs are spectral expanders]

Fact: $\exists G$ w/ $=$, called Ramanujan graphs

Fact: $\exists$ Ramanujan graphs w/ vertex expansion $\le D/2$ [so opt spectral $\not\Rightarrow$ opt vertex]

spectral $\Rightarrow$ vertex [show spectral $\Rightarrow$ vertex]

defn: π prob dist $CP(\pi)$ = collision probability $= \Pr_{X,Y \leftarrow \pi}[X=Y] = \sum_i \pi_i^2$
$= \|\pi\|^2 = \langle \pi, \pi \rangle$

[measure of randomness]

$\mathrm{supp}(\pi) = \{i : \pi_i > 0\}$ [measure of randomness]

eg: $S \subseteq [N]$

$\pi = \mathbb{1}_S/|S|$ uniform on S $\mathrm{supp} = S$

$CP(\pi) = |S| \cdot 1/|S|^2 = 1/|S|$

lem: $CP(\pi) \ge 1/\mathrm{supp}(\pi)$ [equality attained when π uniform over support]

Pf: Recall Cauchy Schwartz $|\langle x,y \rangle|^2 \le \langle x,x \rangle \langle y,y \rangle$

$S = \mathrm{supp}(\pi)$

$1 = |\langle \mathbb{1}_S, \pi \rangle|^2 \le \langle \mathbb{1}_S, \mathbb{1}_S \rangle \langle \pi, \pi \rangle = |\mathrm{supp}(\pi)| \cdot CP(\pi)$ [new robust version of prev]

lem: $CP(\pi) = \|\pi - u\|^2 + 1/N = \|\pi - u\|^2 - CP(u)$

Pf: $\|\pi - u\|^2 = \langle \pi - u, \pi - u \rangle = \underbrace{\langle \pi, \pi \rangle}_{CP(\pi)} - \langle \pi, u \rangle - \langle u, \pi \rangle + \langle u, u \rangle$

$= 1/N$

thm: G D-regular, G is $(N/2, 1+\Omega(1))$ vertex expander if $\lambda(G) \le 1 - \Omega(1)$

Pf: $\exists$ random walk matrix M [M damps $\|\pi - u\|$, so CP goes down, so support goes up]

$S \subseteq V$ $|S| = n/2$ [really need $|S| \le n/2$]

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$$\|M\pi - u\|^2 \le \lambda^2 \underbrace{\|\pi - u\|}_{= CP(\pi) - 1/N = \frac{1}{|S|} - \frac{1}{N} = \frac{2}{N} - \frac{1}{N} = \frac{1}{N}}$$

$$\|M\pi - u\|^2 = \underbrace{CP(M\pi)}_{\approx \frac{1}{\mathrm{supp}(M\pi)}} - 1/N$$

$$\frac{1}{\mathrm{supp}(M\pi)} \le \lambda^2 \frac{1}{N} + \frac{1}{N}$$

$$\mathrm{supp}(M\pi) = |A(S)| \ge \frac{N}{1+\lambda^2} \ge \underbrace{\frac{2}{1+\lambda^2}}_{=1+\Omega(1)} \underbrace{\frac{N}{2}}_{=|S|}$$

Thm. $\forall \delta > 0$ $D \ge 2$ $(M_2, 1+\delta)$ remix $\Rightarrow$ $\gamma(G) \ge \Omega\left(\left(\frac{\delta}{D}\right)^2\right)$ [not pow mixer]

Cor. $\gamma(G) \ge \Omega(1)$ $\equiv$ $(M_2, 1+\Omega(1))$ remix
$\leftarrow D \le O(1)$

today: defn expander
extensions
relations

Admin: ps 1 back Wed
ps 2 due Mon
ps 3 out Wed

next time: more relations
random walks