CS 598 Pseudorandomness: Lecture 8

今天的定义和拓展

**Vertex expander:**

\( G = (V, E) \)

\[ \mathcal{NS} = \{ u : (u \in U \land E \in E) \} \]

- \( |E| \geq \Omega(V) \) (symmetric graph)
- \( \mathcal{NS} \) is an expander
- eg: \( D \) regular \( \log n \)
- well connected

**Definition:**

\( G = (V, E) \) vertex expander if \( V \subset V \) (similar)

- \( |\mathcal{NS}| \geq n - \epsilon \)
- \( \epsilon \) is small

**Properties:**

- \( D \)-regular graph, \( D = 1/\epsilon \)
- \( N = |V| \rightarrow \infty \) is not family 2
- \( A \subset D \), \( KA \subset N \)

**Claim:**

- \( (R(N), \cdot + (D, 1/\epsilon)) \) \( (\mathcal{NS}) \) is vertex expander

**Remarks:**

- Random \( D \)-regular graphs are complicated
- \( \mathcal{NS} \) is not an expander

**Examples:**

- \( A = D-1-\epsilon \) is optimal
- Random \( D \)-regular graphs are complicated
- \( \mathcal{NS} \) is not an expander

**Theorem:**

- \( D = 3 \), \( \mathcal{NS} \) is \( (R(N), \cdot + (D, 1/\epsilon)) \) \( (\mathcal{NS}) \) is vertex expander

**Random Graph:**

- \( D = 2 \) has no expander
- \( A = D-1-\epsilon \) is optimal
- Random \( D \)-regular graphs are complicated
- \( \mathcal{NS} \) is not an expander

**Example:**

- \( D = 3 \), \( \mathcal{NS} \) is \( (R(N), \cdot + (D, 1/\epsilon)) \) \( (\mathcal{NS}) \) is vertex expander

**Conclusion:**

- \( \mathcal{NS} \) is not an expander

**Proof:**

- \( \mathcal{NS} \) is not an expander

**Random Graph:**

- Each vertex in \( L \) chooses \( D \) neighbors in \( R \), independently.
\[ k \leq N, \quad S \subseteq \{1, \ldots, k\} \quad \text{Wawe } S \text{ to expand } V \]

Choose \( N \) such that \( |\text{NCS}| > (N-2)|S| \)

\[ \text{Pr} \left[ |\text{NCS}| > (N-2)|S| \right] = \text{Pr} \left[ z < \frac{k^2}{N} \right] \]

\[ \text{Pr} \left[ |\text{NCS}| > \binom{N}{k} \right] \leq \left( \frac{k^2}{N} \right)^{2k} \left( \frac{k^2}{N} \right)^{2k} \]

\[ \text{Pr} \left[ |\text{NCS}| > \binom{N}{k} \right] \leq \left( \frac{e^{-20k}}{4N} \right)^k \leq \frac{N}{e^k} \]

\( \text{Remark:} - \text{bi-regular vertex expanders via union of } D \)

- Match perfect matchings \( \Pi \) for indegree of \( 1 \)
- Can translate results to uniformly random \( D \)-regular graphs \( \Pi \)

\( Q: \text{explicit vertex expanders?} \) \( \text{help him consider } \text{uniform random expander } \Pi \)

**Question?**

**Spectral expanders:** \( G = (V,E) \) \( D \)-regular, and walk \( \Pi \)

\[ \lambda(G) = \max_{\text{non-constant } \Pi} \frac{\|1\|}{\|\Pi\|} \]

**Spectral gap:** \( \gamma(G) = 1 - \lambda(G) \)

- spectral expander \( \gamma(G) = 1 - \lambda(G) \)

**Note:** \( \gamma(G) \approx D \) \( \text{for constant } D \)

**Edge expander:**

**Def:** \( C \subseteq (V, E) \text{ edge expander if } \gamma_5 \leq \text{cut}(S,S) \)

\[ \text{cut}(S,S) \geq \varepsilon \cdot D \cdot |S| \]

**Examples:**

**Computing sparse cuts:**

- dense community in social networks
- clustering / partitioning
- dynamic programming / limited in vertex between subproblems

\[ \text{cut}(S,S) \geq \varepsilon \cdot D \cdot |S| \]

**Special cases:**

- no better cuts in Japan
relations
Q: edge is vertex is spectra?  Express edge, vertex by spectra.
A: different, yet similar

company: NP-hard

process: can give relationship

yet: \((M_1, M_2)\) vertex expansion \(\Rightarrow A(l)\) spectra expansion
can be discovered by more can be connected is disjoint union of expansions

optimal param:
\[
\text{vertex: } (\infty, N, D - 1, i, i)
\]
\[
\lambda(G) \leq \sqrt{\frac{D - 1}{D}}
\]

Fact: random graphs have \(\lambda(G) \leq \sqrt{\frac{D - 1}{D}} + 0.1\)

Four: \(A(G) = i\), called Ramsey's graph

Five: \(A(Ramsey)\) of vertex expansion \(\leq \frac{1}{2}\)

spectral as vertex is show spectral of vertex

idea: it prob dist \(CP(l)\) collision probability: \(Pr = \sum_{i = 1}^{N} \pi_i^2 \sum_{i = 1}^{N} \pi_i^2 = \sum_{i = 1}^{N} \pi_i^2 \sum_{i = 1}^{N} \pi_i^2 = \sum_{i = 1}^{N} \pi_i^2 \sum_{i = 1}^{N} \pi_i^2 = i(M) \leq <M, T>

\(\text{measure of randomness}\)

eq: \(S \leq (M)
\)

the: \(S/2\) union on \(S\) \(\text{Supp} = S\)
\[
CP(l) = |S| \cdot \frac{1}{|S|} = \frac{1}{|S|}
\]

lem: \(CP(l) \geq \frac{1}{\text{Supp}(l)}\) [equally exists when it union are support]

pf: Recall Cauchy Schwarz \(\langle u, v \rangle^2 \leq \langle u, u \rangle \cdot \langle v, v \rangle \)
\[
\text{Supp}(l) = \{ i : \pi_i > 0 \} \quad \text{[measure of randomness]}
\]

eq: \(S \leq (M)
\)

the: \(A(M)\) union on \(A\) \(\text{Supp} = A\)
\[
CP(l) = |A| \cdot \frac{1}{|A|} \geq \frac{1}{|A|}
\]

lem: \(CP(l) \geq \frac{1}{\text{Supp}(l)}\) [equally exists when it union are support]

pf: \(\text{Cauchy Schwarz}\) \(\langle u, v \rangle^2 \leq \langle u, u \rangle \cdot \langle v, v \rangle \)
\[
\text{Supp}(l) = \{ i : \pi_i > 0 \} \quad \text{[measure of randomness]}
\]

then: \(G - \text{regular}, G \Rightarrow (M_2, 1, 2)\) vertex expansion \(\Rightarrow \lambda(l) < 1 - \frac{1}{2} \langle l \rangle\)

pf: \(\text{Ramsey with many}\) \(M \Sigma M\) changes \(\langle u, v \rangle\), so \(CP\) goes down, so \(\text{Supp}\) goes up
\[
S \leq V \|S\| = \frac{M}{2} \quad \text{[really need } \frac{1}{2} < \sqrt{D} \]
If \( \| M \| = \| u \| \leq \| u - u \| \) then \( \| M \| = \| u \| \leq \| u - u \| \leq \| M \| \leq \| u \| \leq \| u - u \| \).

\[
\frac{1}{\sup(\| M \|)} \leq \frac{\| u \|}{\| u - u \|} \leq \frac{\| u \|}{\| u - u \|} = \frac{\| u \|}{\| u - u \|} = \frac{\| u \|}{\| u - u \|}.
\]

Theorem: \( \forall \alpha > 0 \), \( \exists \beta \) such that \( (1 + \beta^2) \) is not in \( x \) if \( \beta < \beta^2 \).

Corollary: \( \beta(x) \leq \beta(1) = (1 + \beta^2) \) is not in \( x \).

Today: data expand

Relating:

- Add: ps 4 back weird
- ps 2 due Man
- ps 3 of wed

Next: HW: man replace
- Random night